a gravitational analogue

(automated transcription)

Hyperfine interactions appear in the context of atoms, nuclei, electrons, small objects where quantum mechanics is important. And that scares away some people. Therefore, I would like now to develop the formalism of hyperfine interactions in a purely classical mechanics context, namely in a situation where only gravitation works. So no quantum physics appears at all here. You can forget of all complications related to quantum physics. We can focus only on the nature of the hyperfine interactions themselves. Our model system to do this will be two mass distributions. You see here at the left mass distribution 1, a continuous mass distribution, and the vector r1 probes any point inside this mass distribution. The other mass distribution is m2 and the vector r2 probes any point inside mass distribution 2. We describe these vectors and these mass distributions into an axis system. That axis system is given here. And for convenience, we will fix this axis system. We will fix the origin of this axis system in the center of mass of mass distribution 1. The vector r0 is a vector that connects the origin of the axis system with the center of mass of mass distribution 2. We will now ask what is the gravitational potential energy of this system, where we assume that both mass distributions are completely static. Everything is moving. Everything is frozen. We can do this in two ways. We can calculate the gravitational potential field of mass distribution 1 and then ask what is the energy of mass distribution 2 in that field. Or we can calculate the gravitational field of mass distribution 2 and ask what is the potential energy of mass distribution 1 in that field. And let's do it in that second way. Let us focus at a particular position r1 inside mass distribution 1. What is the field due to mass distribution 2 at that point? Well, so far we just write it as v of 2 at position r1. Which mass is present at that position? That's an infinitesimal part of mass distribution 1. So that's the mass density of mass distribution 1, rho 1, at that point r1 multiplied by an infinitesimal volume element dr1. That infinitesimal portion of mass times the local field by mass distribution 2. That is the contribution to the potential energy of that infinitesimal part of mass distribution 1. If we now integrate that over the entire mass distribution 1, we have the total potential energy. And that's what the first expression of this slide tells us. So we have now to specify how that v2 looks like. And that's a straightforward application of Newton's law. The gravitational potential due to mass distribution 2 at position r1 is an infinitesimal volume element of mass distribution 2, which is rho 2 times dr2, divided by the distance between the point r2 and the point r1. And this integrated over mass distribution 2 and multiplied by the gravitational constant. If we plug in that second equation into the first one, we get the equation here at the bottom, which gives you the potential energy corresponding to the gravitational interaction of these two mass distributions. And you clearly see that if you know how these mass distributions look like, if you know the expressions for rho 1 and rho 2, you have everything you need to work out this double integral and to find that gravitational potential energy. Of course, if these mass distributions are really general, if these shapes are weird, it will not be so easy to work out that double integral. That's a complication that physicists faced already a long time ago and they found a solution for that, or rather mathematicians found a solution for that. You can expand the division by the distance, this 1 over r2 minus r1. You can expand this according to what is given here the Laplace expansion. So this particular function can be written as a sum over n and q, where n is the order of spherical harmonics and q are components of spherical harmonics. So a sum over n and q of a product between spherical harmonics and a factor in front of that, which depends on distances inside mass distribution 1 and 2. The peculiar feature of this Laplace expansion are these symbols r smaller than and r larger than. So let's go back to the previous slide, where you see that this 1 over r2 minus r1 appears inside that double integral, so depending on where you are in the double integration, these vectors r2 and r1 change. For any step in that integration process, you always have to wonder which of the two vectors, r1 or r2, is the smallest of the two. The smallest of the two becomes r smaller than, the larger of the two becomes r larger than, and that assignment is not fixed. Depending on where you are in the integration process, the role of r1 and r2 can change. So that's a complication we have to keep in mind, but let's assume for a while that we can deal with that. So now you have this expansion, you plug that in into the double integral and you get the equation at the end of this slide here. Let's now make an assumption, because so far we haven't gained anything. Our double integral is still as complicated as it was before. Let us make an assumption and let us consider only mass distributions that are such that every vector r1 is smaller than any vector r2. That is a condition that is fulfilled by the cartoon where we started from. Any vector r1 inside that first mass distribution is smaller than any of the vectors r2, even smaller than the smallest vector r2 that you can have. I draw a similar situation here in the lower right corner, a situation where the cartoon looks more like a nucleus interacting with an electron cloud, a nucleus in red with some general shape and a vector r1, which is called here a vector r, that probes the volume inside the nucleus, and the electron cloud in blue where we have the vector r2 or here a vector r' that probes any point inside the electron cloud. So also in this second cartoon here, the condition that every r is smaller than any r1 is fulfilled. So for a situation where this assumption holds, we will continue with our double integration. If that condition holds, then we can uniquely assign the vector r1 to the symbol r smaller than, and the vector r2 to the vector r larger than. If we do that, our double integration over that sum can be split up in a specific way. We can group all terms that depend on mass distribution 1, and we can group all terms that depend on mass distribution 2, and there is nothing in between that depends on mass distribution 1 and 2. So we can write this summation as a sum over still the orders of the spherical harmonics and their components n and q, and every time, in every term of the sum, there will be a product between an object q, which behaves as a spherical harmonic, it has order n component q, an object q that depends only on properties of mass distribution 1, you have the full expression for a general component of q here, and there are only properties related to mass distribution 1 here, and that is multiplied by something that depends only on mass distribution 2. So you have a full separation of the two objects, no cross terms between them. But that only holds if every vector r1 is smaller than any possible vector r2. Let's look at the lowest order terms in that infinite summation. So the 0th order term, the 0,0 term, looks like this. You have the object q,0,0 complex conjugate times v,0,0, and there is only a 0,0 component. These objects q,0,0 and v,0,0, they are scalar properties. Two scalar properties that are multiplied to give what we call the monopole contribution to the gravitational potential energy. We have the full expressions of this q,0,0 and v,0,0 on the previous slide, so just fill out n equals 0, q equals 0 here, and you will find expressions that are fairly straightforward to deal with, and you will recognize in this way that q,0,0 is a number that is nothing else than the total mass of mass distribution 1. While v,0,0, if you look at that expression there, that tells you what is the gravitational potential by mass distribution 2 at the origin of the axis system, and therefore at the center of mass of mass distribution 1. That would be the gravitational interaction between two point masses, which is obviously the meaning of the monopole approximation. Let's go one term further in the sum, and then we enter the what we call the dipole term. Now we have a product between an object q that has three components, q with superscript 1 can have components minus 1, 0 or plus 1, and the general expression for these is given here as first expression on this slide. So that is an object that is a vector, three components, that's a vector. The physical meaning of that object, the dipole moment, the gravitational dipole moment, is that it gives the position in space of the center of mass of mass distribution 1 with respect to the origin of the axis system. And you can see already here that as we have taken mass distribution 1, as we have taken the axis system to have its origin in the center of mass of mass distribution 1, this dipole moment vector will be 0 in our case. The dipole field is also a vector, we will look at its meaning soon, and the dot product between both, what you see here, the third equation is an expression for a dot product between two vectors in spherical notation. The dot product between these two vectors is the dipole contribution to the gravitational energy. Let's look in more detail to the meaning of this dipole moment vector and dipole field vector. We have these objects Q1 with subscripts plus 1, minus 1, or 0. If we group them in the way that is written there on top of this slide, you get an object which temporarily we will call Qx. And if you now fill out the expressions for Q1 minus 1 and Q1 plus 1, you will find the next equation, which turns out to be the integral over mass distribution 1 times x, the x-component of the vector r1. And you might recognize in this the x-component of the position vector of the center of mass of mass distribution 1. And in the same way you can find the y and z-components. So indeed, these Q objects, they are related to the position vector of the center of mass. And in the same way, you could write the Vx, Vy, Vz, which is the opposite of the gravitational field of mass distribution 2 at the origin. So which gravitational field does mass distribution 2 produce at the origin of the axis system, so at the center of mass of mass distribution 1. That's the dipole term, and as we have seen that by our choice of the origin of the axis system, the position vector of the center of mass of mass distribution 1 is 0. The dipole term will not contribute to our gravitational potential energy. So the next term that will contribute will be the quadrupole term. And here the Q object is now an object of order 2 with 5 components, plus 2, plus 1, 0, minus 1, minus 2, so a spherical tensor of rank 2. That will be the quadrupole moment of mass distribution 1. It will express how mass distribution 1 deviates from spherical symmetry. And that tensor of rank 2 will be multiplied with another tensor of rank 2, which is the quadrupole field of mass distribution 2 at the origin of the axis system. And that can be interpreted as the gradient of the electric field by, sorry, not the gradient of the electric field, because we are working in a gravitational problem here, the gradient of the gravitational field of mass distribution 2 at the origin of our axis system. A dot product between these two vectors, between these two tensors of rank 2, gives us a scalar quantity, which is the quadrupole contribution to the potential energy. Let's zoom in on this quadrupole moment and quadrupole field for a while. So the quadrupole moment, if you make the full derivation, can be written in this way. An integral over mass distribution 1 of the mass density distribution rho 1 multiplied by this symmetric and traceless matrix. You can take it as an exercise to show that this matrix is indeed traceless. The interpretation of this quadrupole moment, which you might have encountered in classical mechanics courses before, is such that if you consider this in an axis system, which is a principal axis system, an axis system adapted to the symmetry of the problem, in such a principal axis system, this quadrupole moment matrix will become a diagonal matrix. And the largest component on the diagonal, that is conventionally called the z-component. So if that z-component is 0, you have a spherical mass distribution. If it is positive, you have an elongated mass distribution. And if it is negative, you have a squeezed mass distribution. The quadrupole moment tensor. The quadrupole field. The full expression can be shown to be like the first equation on this slide.

Again a symmetric and traceless tensor. Again easy to show that it is traceless. And this expression is the one which you would have found if you have applied this Laplace expansion. Now there is an alternative way of making the same expansion. Not using this Laplace version with spherical harmonics, but something that looks like a Taylor expansion, but now for a function on a vector domain. Usually you use Taylor expansions in one dimension. You can do this also for vector functions. It's a little bit more involved, but it's the same concept. Now let's assume that we would have done this expansion using a Taylor expansion. Then we would have found for the quadrupole field tensor this 3 by 3 matrix. A matrix made by second derivatives with respect to the x, y, and z components of the gravitational potential by mass distribution 2 at the origin of our axis system. Is that a tensor with the same properties as the first on this slide? So yes, it's also symmetric. The order of differentiation doesn't matter. Is it traceless? Well let's make the trace and that's the last equation on this slide. The trace is the Poisson equation and for the Poisson equation we know this general relation. The Laplacian of the potential at the origin is proportional to the density of that same mass distribution at the origin. Well we had assumed that any vector r1 was smaller than any vector r2, which means that it is not possible for mass distribution 2 to be present at the origin of the axis system. So rho 2 at the origin of the axis system is zero and indeed so this Laplacian is zero. This matrix is traceless. We have all the properties we wanted. So here you see for those who are interested in the mathematical details how this Taylor expansion for a function on a vector domain looks like and it's with this expression b2 as it is indicated here that you can make this Taylor expansion of the gravitational potential energy. Imagine now that we would have made this spherical harmonics expansion, Laplace expansion and this Taylor expansion. In both cases we have monopole, dipole, quadrupole and so on terms only they do not look exactly identical. If you look at the details the monopole and the dipole terms will turn out to be completely identical and the difference is only in the quadrupole term. The first difference is in the quadrupole term. Therefore I write here this potential energy which is a dot product between the left object which is now the quadrupole tensor but found from the Taylor expansion with the gravitational field gradient the quadrupole field again found from the Taylor expansion. What you see here is not a matrix product because then the result would be a 3 by 3 matrix. It's really a dot product and the convention for such a dot product between such objects is that you multiply the 1,1 object with the 1,1 object of the second matrix plus the 1,2 object with the 1,2 component of the second matrix plus 1,3 times 1,3 and so on. So not a matrix multiplication but really a dot product. You can see that the quadrupole moment tensor as it is given here is not really a spherical tensor because it's not traceless. If you would calculate the trace here there is no reason why that object would be zero. We could make it traceless by subtracting 1,3 of the trace from every diagonal component and adding that back in at the end. So that's a decomposition of that Cartesian object into something that has the properties of a quadrupole moment and something else. The same you could do for the gravitational field gradient tensor. We had there the trace which was our Poisson equation and which was in general not necessarily zero, only zero if we are in the situation that is allowed by our assumption. So let's assume this is now more general and let's subtract 1,3 of the trace, so 1,3 of the Laplacian of V2 at the origin and add that back again. So both these Cartesian tensors have been written as a sum of a spherical tensor plus something else. That is what this slide leads to. And now we make the dot product between that first sum of two tensors with that second sum of two tensors. And if you work it out you find first the dot product between the two spherical tensors which is what we found also after the treatment with the Laplace expansion. A few of the cross terms are zero and the other non-zero part is this dot product between two diagonal matrices which can be simplified up to the form you find here at the bottom. The mass density of mass distribution 2 at the origin of the axis system multiplied by something that you could interpret as a mean square radius of mass distribution 1. If our initial assumption that any vector r1 is smaller than any vector r2 is fulfilled then rho2 at zero must be zero and this term does not contribute. But if that condition would not be fulfilled, so if we would be in the more general case, that term survives. So here you see that the Cartesian expansion, the Taylor expansion is more general than the spherical expansion, the Laplace expansion with our assumption. The more general treatment leads to a correction term that depends on the mass density of mass distribution 2 at the origin. All of this can be summarized in this table. So let's first look at the first half which is the situation where there is no overlap between mass distribution 2 and mass distribution 1. With no overlap I mean this condition of r1 smaller than r2 is fulfilled. In that case we have monopole, dipole and quadrupole terms which always consist of a dot product between a property of mass distribution 1 with a property of mass distribution 2. In the monopole term this is a dot product between two scalars. In the dipole term this is a dot product between two vectors. In the quadrupole term it is a dot product between two tensors of rank 2. In the monopole term the monopole moment of mass distribution 1 is the mass of that mass distribution. The dipole moment of mass distribution 1 is the position vector of the center of mass of mass distribution 1 which was the zero vector. And the quadrupole moment of mass distribution 1 describes the deviation from spherical symmetry. The monopole field of mass distribution 2 is the gravitational potential by that mass distribution at the origin of the axis system. The monopole field of mass distribution 2 is the gravitational potential by mass distribution 2 at the origin. The dipole field of mass distribution 2 is the opposite of the gravitational field by mass distribution 2 at the origin. And the quadrupole field of mass distribution 2 is the gradient of the gravitational field by mass distribution 2 at the origin. If you leave that particular assumption of R1 smaller than R2, you go to the situation with overlap which you can find more straightforwardly when you use a Taylor expansion of the 1 over R function. And here you find that the most important correction due to that overlap is a term that depends both on the size of mass distribution 1, its mean square radius, and the amount of mass of mass distribution 2 that is present at the origin of the axis system.