the double ring example

(automated transcription)

As an illustration for the gravitational analog for hyperfine interactions, we will now look at the example of a double ring. What is that specific case? Well, we have our mass distribution 1, which in this example is a dumbbell. A dumbbell of two point masses connected by a massless rod. Each of these two point masses has mass m1 divided by 2, so the total mass of this dumbbell is m1. The distance between the two point masses is l1. We put that dumbbell such that its center of mass, which is halfway on the rod, coincides with the origin of our axis system. Because we are working towards the case of nuclei interacting with electrons, that dumbbell will be our analog of a nucleus. But here, so it's still a gravitational example. So this nucleus is interacting with an electron cloud, but what is our analog for the electron cloud? We take here two rings, so these are not dishes, but really rings, with a mass m2. The total mass of the two rings is m2. Each ring has a total mass m2 over 2. The radius of the rings we call capital R, and the distance between the rings is h. So the two rings are parallel to the xy-plane. So we freeze values for all of these quantities l1, R, and h, m1 and m2, and just as in the general case, we ask what is the total gravitational energy corresponding to that particular situation. In this example, you could find the exact solution, but we will use the Taylor expansion in order to find the approximate solution. When does this expansion converge quickly? In the general case, we saw it converges quickly when R1 over R2 is much smaller than 1. And in this particular case, you can convert this to the quantities we have defined in the double ring system. That means basically that l1 must be a much smaller distance than h and R. So if we have a very tiny dumbbell and big rings that are quite far apart from each other, then the Taylor expansion will very quickly converge to the exact solution, which means we can stop after the quadrupole term. Well, let's work that out. It's quite straightforward to find that the monopole contribution to the potential energy is this expression here. The dipole contribution will be zero, as in the general case, so the next correction to that is the quadrupole contribution. And our Cartesian version of the quadrupole moment tensor for the dumbbell, for the nucleus, looks like this. The quadrupole field due to the double ring looks like this. So in both cases, symmetric 3 by 3 matrices, traceless. And in the case of the quadrupole field of the double ring, it's even a diagonal matrix. So that means that the axis system, which we have drawn on the previous slide, that this axis system is a principal axis system for this double ring system. By multiplying element by element these two 3 by 3 matrices and summing everything, we find the quadrupole contribution to the gravitational potential energy, and that is the expression which you see here. It's more instructive if we draw a specific example of this. So I draw here the quadrupole correction to the gravitational energy as a function of the angle theta, and the angle theta is the angle that gives you the orientation of the dumbbell with respect to the z-axis. So if theta equals zero, then the dumbbell lies parallel to the z-axis. If theta equals 90 degrees, the dumbbell lies in the xy-plane. And what do you see? On the picture, the quadrupole correction to the gravitational energy is most negative when the dumbbell is parallel to the z-axis, and is most positive when the dumbbell lies in the xy-plane. So the lowest energy configuration of this system up to the quadrupole interaction is a situation where the dumbbell is along the z-axis, at least if that pre-factor alpha here, which depends on the geometry of the system, is positive. If alpha would have the other sign, which is also possible, then the situation reverses. And that is what is sketched here. So when is alpha positive? If you inspect this expression, you see that alpha is positive if the separation between the rings is much larger than the radius of the rings. So that's the left case. Alpha is negative if the radius of the rings is much larger than the separation. And alpha is zero in the particular case where the separation between the rings, h, is square root of two times the radius of the rings. So in that case, if alpha is zero, then whatever is the orientation of the dumbbell, there is no quadrupole contribution to the gravitational potential energy. You can draw the previous picture also in a form that more looks like an energy level diagram. So on the left of the picture, you have the monopole contribution to the total energy. And on the right, the quadrupole correction to it. So for some orientations, there will be a positive energy contribution. For other orientations, a negative energy contribution. And everything that is in between these two extremes is possible. So depending on how the dumbbell is oriented, you will have an energy that is somewhere inside this gray region. Hence to conclude, if you want to visualize the somewhat abstract multipole expansion of a gravitational problem as we have seen before, if you want to visualize that with a specific example, it's useful to think about this dumbbell and the double ring.