gravitational analogue

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1

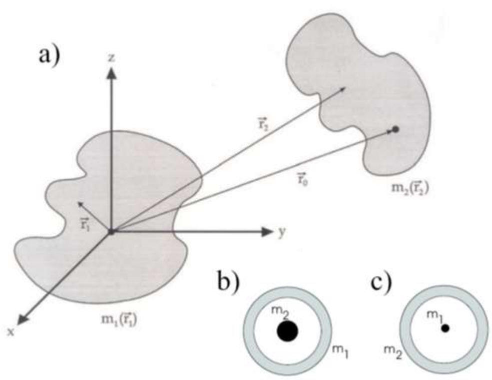


Fig. 2.1. *Three situations of two mass distributions interacting by gravitation.*

2

Gravitational potential energy of this system =

- gravitational potential energy of m_2 in the field of m_1

or

- **gravitational potential energy of m_1 in the field of m_2**

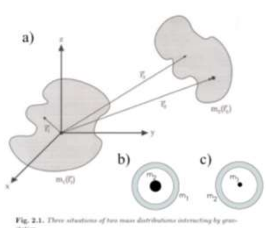
$$E_{pot} = \int_1 \rho_1(\boldsymbol{r}_1)\, V_2(\boldsymbol{r}_1)\, d\boldsymbol{r}_1$$

Potential field of m_2 at position r_1 :

$$V_2(\boldsymbol{r}_1) = -G \int_2 \frac{\rho_2(\boldsymbol{r}_2)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|}\, d\boldsymbol{r}_2$$

And hence :

$$E_{pot} = -G \int_1 \int_2 \frac{\rho_1(\boldsymbol{r}_1)\rho_2(\boldsymbol{r}_2)}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|}\, d\boldsymbol{r}_1\, d\boldsymbol{r}_2$$



3

How to treat the double integral? ➔ Laplace expansion

$$\frac{1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|} = 4\pi \sum_{n,q} \frac{r_<^n}{r_>^{n+1}} \frac{1}{2n+1} Y_q^{n*}(\theta_1, \phi_1) Y_q^n(\theta_2, \phi_2)$$

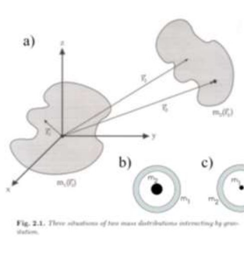
$$r_< = min\,(r_1, r_2)$$

$$r_> = max\,(r_1, r_2)$$

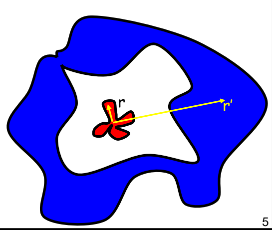
$$E_{pot} = -4\pi G \int_1 \int_2 \rho_1(\boldsymbol{r}_1)\rho_2(\boldsymbol{r}_2) \left(\sum_{n,q} \frac{r_<^n}{r_>^{n+1}} \frac{1}{2n+1} Y_q^{n*}(\theta_1, \phi_1) Y_q^n(\theta_2, \phi_2) \right) d\boldsymbol{r}_1\, d\boldsymbol{r}_2$$

4

Assumption: the mass distributions are such that any r_1 is smaller than any r_2



Imagine an example where this is not fulfilled…



5

Condition: consider only cases for which any r_1 is smaller than any r_2

Consequence: $r_< = r_1$
$r_> = r_2$

$$E_{pot} = -4\pi G \int_1 \int_2 \rho_1(\boldsymbol{r}_1)\rho_2(\boldsymbol{r}_2) \left(\sum_{n,q} \frac{r_<^n}{r_>^{n+1}} \frac{1}{2n+1} Y_q^{n*}(\theta_1, \phi_1) Y_q^n(\theta_2, \phi_2) \right) d\boldsymbol{r}_1\, d\boldsymbol{r}_2$$

$$E_{pot} = \sum_{n,q} Q_q^{n*} V_q^n$$

$$Q_q^n = \sqrt{\frac{4\pi}{2n+1}} \int_1 \rho_1(\boldsymbol{r}_1)\, r_1^n\, Y_q^n(\theta_1, \phi_1)\, d\boldsymbol{r}_1$$

$$V_q^n = -G \sqrt{\frac{4\pi}{2n+1}} \int_2 \frac{\rho_2(\boldsymbol{r}_2)}{r_2^{n+1}}\, Y_q^n(\theta_2, \phi_2)\, d\boldsymbol{r}_2$$

6

Discussion :

monopole term

monopole moment	$Q_0^0 = m_1$	scalar
monopole field	$V_0^0 = -G \int \frac{\rho_2(\boldsymbol{r}_2)}{\lvert \boldsymbol{r}_2 \rvert} \, d\boldsymbol{r}_2$	scalar
monopole energy	$E_{pot}^{(0)} = Q_0^{0*} V_0^0$	dot product ➔ scalar

7

7

Discussion :

dipole term

dipole moment	$Q_q^1 = \sqrt{\frac{4\pi}{3}} \int_1 \rho_1(\boldsymbol{r}_1)\, r_1\, Y_q^1(\theta_1, \phi_1)\, d\boldsymbol{r}_1$	vector
dipole field	$V_q^1 = -G \sqrt{\frac{4\pi}{3}} \int_2 \frac{\rho_2(\boldsymbol{r}_2)}{r_2^2}\, Y_q^1(\theta_2, \phi_2)\, d\boldsymbol{r}_2$	vector
dipole energy	$E_{pot}^{(1)} = \sum_{q=-1,0,1} Q_q^{1*} V_q^1$	dot product ➔ scalar

8

8

interpretation of the dipole moment of m_1:

$$Q_x = \frac{\sqrt{2}}{2}\left(Q_{-1}^1 - Q_{+1}^1\right) = \int_1 \rho_1(\boldsymbol{r}_1)\, r_1 \sin\theta \cos\phi \, d\boldsymbol{r}_1 = \int_1 \rho_1(\boldsymbol{r}_1)\, x_1 \, d\boldsymbol{r}_1$$

$$Q_y = \int_1 \rho_1(\boldsymbol{r}_1)\, y_1 \, d\boldsymbol{r}_1$$

$$Q_z = \int_1 \rho_1(\boldsymbol{r}_1)\, z_1 \, d\boldsymbol{r}_1$$

position vector of center of mass of m_1

interpretation of the dipole field by m_2:

$$V_x = -G \int_2 \frac{\rho_2(\boldsymbol{r}_2)}{\lvert r_2 \rvert^3}\, x_2 \, d\boldsymbol{r}_2$$

$$V_y = -G \int_2 \frac{\rho_2(\boldsymbol{r}_2)}{\lvert r_2 \rvert^3}\, y_2 \, d\boldsymbol{r}_2$$

$$V_z = -G \int_2 \frac{\rho_2(\boldsymbol{r}_2)}{\lvert r_2 \rvert^3}\, z_2 \, d\boldsymbol{r}_2$$

opposite of the gravitational field by m_2 at the origin

9

9

Discussion :

quadrupole term

quadrupole moment	$Q_q^2 = \sqrt{\frac{4\pi}{5}} \int_1 \rho_1(\boldsymbol{r}_1)\, r_1^2\, Y_q^2(\theta_1, \phi_1)\, d\boldsymbol{r}_1$	tensor
quadrupole field	$V_q^2 = -G \sqrt{\frac{4\pi}{5}} \int_2 \frac{\rho_2(\boldsymbol{r}_2)}{r_2^3}\, Y_q^2(\theta_2, \phi_2)\, d\boldsymbol{r}_2$	tensor
quadrupole energy	$E_{pot}^{(2)} = \sum_{q=-2,\ldots,2} Q_q^{2*} V_q^2$	dot product ➔ scalar

10

10

quadrupole moment :

$${}_cQ_{sh}^{(2)} = \int_1 \rho_1(\boldsymbol{r}_1) \begin{bmatrix} 3x_1^2 - r_1^2 & 3x_1y_1 & 3x_1z_1 \\ 3x_1y_1 & 3y_1^2 - r_1^2 & 3y_1z_1 \\ 3x_1z_1 & 3y_1z_1 & 3z_1^2 - r_1^2 \end{bmatrix} d\boldsymbol{r}_1$$

- symmetric
- trace-less (show)

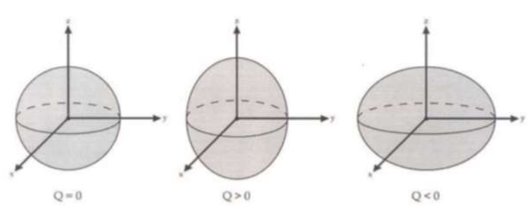


Fig. 2.2. *A spherical, prolate and oblate mass distribution (with respect to the z-axis).*

11

11

quadrupole field :

$${}_cV_{sh}^{(2)} = -G \int_2 \frac{\rho_2(\boldsymbol{r}_2)}{\lvert \boldsymbol{r}_2 \rvert^5} \begin{bmatrix} 3x_2^2 - r_2^2 & 3x_2y_2 & 3x_2z_2 \\ 3x_2y_2 & 3y_2^2 - r_2^2 & 3y_2z_2 \\ 3x_2z_2 & 3y_2z_2 & 3z_2^2 - r_2^2 \end{bmatrix} d\boldsymbol{r}_2$$

- symmetric
- trace-less (show)

$$V_{ij} = \begin{bmatrix} \frac{\partial^2 V_2(\vec{0})}{\partial x_2^2} & \frac{\partial^2 V_2(\vec{0})}{\partial y_2 \partial x_2} & \frac{\partial^2 V_2(\vec{0})}{\partial z_2 \partial x_2} \\ \frac{\partial^2 V_2(\vec{0})}{\partial x_2 \partial y_2} & \frac{\partial^2 V_2(\vec{0})}{\partial y_2^2} & \frac{\partial^2 V_2(\vec{0})}{\partial z_2 \partial y_2} \\ \frac{\partial^2 V_2(\vec{0})}{\partial x_2 \partial z_2} & \frac{\partial^2 f(\vec{0})}{\partial y_2 \partial z_2} & \frac{\partial^2 V_2(\vec{0})}{\partial z_2^2} \end{bmatrix}$$

The meaning of this tensor is more clear when deriving it using a cartesian Taylor expansion:

➔ (gravitational) field gradient tensor

Traceless?

$$\Delta V_2(\mathbf{0}) = \frac{\partial^2 V_2(\mathbf{0})}{\partial x^2} + \frac{\partial^2 V_2(\mathbf{0})}{\partial y^2} + \frac{\partial^2 V_2(\mathbf{0})}{\partial z^2} = 4\pi\, G\, \rho_2(\mathbf{0})$$

Poisson equation

12

12

Taylor expansion of a scalar function defined on a vector domain:

Consider a function

$$f(\boldsymbol{r}) = \int \frac{g(\boldsymbol{r}_v)}{|\boldsymbol{r}_v - \boldsymbol{r}|}\, d\boldsymbol{r}_v \qquad \text{(B.1)}$$

The integral runs over that part of space where $g(\boldsymbol{r}_v)$ is not zero, which might be a finite or infinite region. If g is a charge or mass distribution, f gives the electric or gravitational potential in a point $\boldsymbol{r}$ (apart from an appropriate factor). That point can be either inside or outside the non-zero region of g (Fig. B-1***). If it lies inside, the denominator in the integral becomes zero and we have to care about the convergence of the integral. The latter is determined by the properties of g. We assume that we know the value of f and of all its derivatives *at the origin* $\mathbf{0}$. What we want to know is the value of f at points $\boldsymbol{r} = (x, y, z)$ that are not far away from $\mathbf{0}$. This means we need a Taylor expansion of $f(\boldsymbol{r})$ around $\mathbf{0}$. The general form of a Taylor expansion around $\mathbf{0}$ for a function with vectors as argument, is:

$$f(\mathbf{0} + \boldsymbol{r}) = \sum_{j=0}^{\infty} \left[\frac{1}{j!} (\boldsymbol{r} \cdot \boldsymbol{\nabla}_{\boldsymbol{r}'})^j\, f(\boldsymbol{r}') \right]_{\boldsymbol{r}'=\mathbf{0}} \qquad \text{(B.2)}$$

dot product between scalars, vectors, tensors,...

13

13

- monopole and dipole terms from Taylor expansion are identical to the Laplace expansion
- difference in the quadrupole term:

$$E^{(2)}_{pot} = \frac{1}{2} \begin{bmatrix} \int \rho_1(\boldsymbol{r}_1) x_1^2\, d\boldsymbol{r}_1 & \int \rho_1(\boldsymbol{r}_1) x_1 y_1\, d\boldsymbol{r}_1 & \int \rho_1(\boldsymbol{r}_1) x_1 z_1\, d\boldsymbol{r}_1 \\ \int \rho_1(\boldsymbol{r}_1) y_1 x_1\, d\boldsymbol{r}_1 & \int \rho_1(\boldsymbol{r}_1) y_1^2\, d\boldsymbol{r}_1 & \int \rho_1(\boldsymbol{r}_1) y_1 z_1\, d\boldsymbol{r}_1 \\ \int \rho_1(\boldsymbol{r}_1) z_1 x_1\, d\boldsymbol{r}_1 & \int \rho_1(\boldsymbol{r}_1) z_1 y_1\, d\boldsymbol{r}_1 & \int \rho_1(\boldsymbol{r}_1) z_1^2\, d\boldsymbol{r}_1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial^2 V_2(\vec{0})}{\partial x_2^2} & \frac{\partial^2 V_2(\vec{0})}{\partial y_2 \partial x_2} & \frac{\partial^2 V_2(\vec{0})}{\partial z_2 \partial x_2} \\ \frac{\partial^2 V_2(\vec{0})}{\partial x_2 \partial y_2} & \frac{\partial^2 V_2(\vec{0})}{\partial y_2^2} & \frac{\partial^2 V_2(\vec{0})}{\partial z_2 \partial y_2} \\ \frac{\partial^2 V_2(\vec{0})}{\partial x_2 \partial z_2} & \frac{\partial^2 f(\vec{0})}{\partial y_2 \partial z_2} & \frac{\partial^2 V_2(\vec{0})}{\partial z_2^2} \end{bmatrix}$$

$${}_cK^{(2)} = \frac{1}{3} \begin{bmatrix} \{3x_1^2\} - \{r_1^2\} & \{3x_1y_1\} & \{3x_1z_1\} \\ \{3y_1x_1\} & \{3y_1^2\} - \{r_1^2\} & \{3y_1z_1\} \\ \{3z_1x_1\} & \{3z_1y_1\} & \{3z_1^2\} - \{r_1^2\} \end{bmatrix} + \frac{1}{3} \begin{bmatrix} \{r_1^2\} & 0 & 0 \\ 0 & \{r_1^2\} & 0 \\ 0 & 0 & \{r_1^2\} \end{bmatrix} \qquad (2.58)$$

$${}_cW^{(2)} = \begin{bmatrix} \frac{\partial^2 V_2(\mathbf{0})}{\partial x_1^2} - \frac{\Delta V_2(\mathbf{0})}{3} & \frac{\partial^2 V_2(\mathbf{0})}{\partial y_1 \partial x_1} & \frac{\partial^2 V_2(\mathbf{0})}{\partial z_1 \partial x_1} \\ \frac{\partial^2 V_2(\mathbf{0})}{\partial x_1 \partial y_1} & \frac{\partial^2 V_2(\mathbf{0})}{\partial y_1^2} - \frac{\Delta V_2(\mathbf{0})}{3} & \frac{\partial^2 V_2(\mathbf{0})}{\partial z_1 \partial y_1} \\ \frac{\partial^2 V_2(\mathbf{0})}{\partial x_1 \partial z_1} & \frac{\partial^2 f(\mathbf{0})}{\partial y_1 \partial z_1} & \frac{\partial^2 V_2(\mathbf{0})}{\partial z_1^2} - \frac{\Delta V_2(\mathbf{0})}{3} \end{bmatrix} + \begin{bmatrix} \frac{\Delta V_2(\mathbf{0})}{3} & 0 & 0 \\ 0 & \frac{\Delta V_2(\mathbf{0})}{3} & 0 \\ 0 & 0 & \frac{\Delta V_2(\mathbf{0})}{3} \end{bmatrix} \qquad (2.59)$$

14

14

$$E^{(2)}_{pot} = \frac{1}{6} \begin{bmatrix} \{3x_1^2\} - \{r_1^2\} & \{3x_1y_1\} & \{3x_1z_1\} \\ \{3y_1x_1\} & \{3y_1^2\} - \{r_1^2\} & \{3y_1z_1\} \\ \{3z_1x_1\} & \{3z_1y_1\} & \{3z_1^2\} - \{r_1^2\} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial^2 V_2(\mathbf{0})}{\partial x_1^2} - \frac{\Delta V_2(\mathbf{0})}{3} & \frac{\partial^2 V_2(\mathbf{0})}{\partial y_1 \partial x_1} & \frac{\partial^2 V_2(\mathbf{0})}{\partial z_1 \partial x_1} \\ \frac{\partial^2 V_2(\mathbf{0})}{\partial x_1 \partial y_1} & \frac{\partial^2 V_2(\mathbf{0})}{\partial y_1^2} - \frac{\Delta V_2(\mathbf{0})}{3} & \frac{\partial^2 V_2(\mathbf{0})}{\partial z_1 \partial y_1} \\ \frac{\partial^2 V_2(\mathbf{0})}{\partial x_1 \partial z_1} & \frac{\partial^2 f(\mathbf{0})}{\partial y_1 \partial z_1} & \frac{\partial^2 V_2(\mathbf{0})}{\partial z_1^2} - \frac{\Delta V_2(\mathbf{0})}{3} \end{bmatrix} + \frac{1}{6} \begin{bmatrix} \{r_1^2\} & 0 & 0 \\ 0 & \{r_1^2\} & 0 \\ 0 & 0 & \{r_1^2\} \end{bmatrix} \cdot \begin{bmatrix} \frac{\Delta V_2(\mathbf{0})}{3} & 0 & 0 \\ 0 & \frac{\Delta V_2(\mathbf{0})}{3} & 0 \\ 0 & 0 & \frac{\Delta V_2(\mathbf{0})}{3} \end{bmatrix}$$

monopole shift :

$$\frac{1}{6}\, {}_cQ^{(0)}_{sz} \cdot {}_cV^{(0)}_{sz} = \frac{1}{6} \Delta V_2(\mathbf{0}) \left\langle r_1^2 \right\rangle = \frac{4\pi G}{6} \rho_2(\mathbf{0}) \int \rho_1(\boldsymbol{r}_1)\, r_1^2\, d\boldsymbol{r}_1$$

only if m_2 extends up to the origin !
(i.e. impossible if always $r_1 < r_2$ ➔ this was a more general derivation)

15

15

no overlap	monopole term	dipole term	quadrupole term
m_1	mass of m_1	position vector center of mass of m_1	quadrupole moment of m_1
m_2	potential by m_2 at origin	opposite of field by m_2 at origin	gradient of gravitional field by m_2 at origin
with overlap			
	correction depending on the size of m_1 and the mass contribution of m_2 at the origin		

16

16